A Tauberian theorem for Cesaro and Abel summability

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1. In this paper we shall be concerned with the well known $(C, 1)$ method of limiting a sequence, viz. taking the limit of $\{t_n\}$:

$$t_n = \frac{1}{n} \sum_{k=1}^{\infty} s_k;$$

we shall also be concerned with Abel's method defined by

$$\lim_{x \to 1-} (1-x) \sum_{k=1}^{\infty} s_k x^k.$$

For both of these methods membership of the set E is a Tauberian condition, where

$$E = \left\{ s : s_n - s_{n-1} = O\left(\frac{1}{n}\right)\right\}.$$

This condition is in many ways the best possible, and certainly cannot be improved from $O(1/n)$ to $O(f(n)/n)$, $f(n) \uparrow \infty$, see (2). However we shall show in this paper that using the topology generated by the sup norm, i.e.

$$\|s\| = \sup_n |s_n|,$$

membership in the closure $\bar{E}$ of E is a Tauberian condition for both these methods.

We first prove:

THEOREM 1. If s is limited by the Abel method and $s \in \bar{E}$, then s is bounded.

PROOF: We suppose $\|s-\hat{s}\|<\varepsilon$ where $\hat{s} \in E$; let

$$f(x)=(1-x) \sum_{k=1}^{\infty} s_k x^k$$

with $\hat{f}(x)$ the function corresponding to $\hat{s}$. Then

$$|f(x)-\hat{f}(x)| \leqq (1-x) \sum_{k=1}^{\infty} |s_k-\hat{s}_k| \, |x|^k < \varepsilon, \quad (O \leqq x < 1).$$

Hence, if $\lim_{x\to 1^-} f(x)$ exists, $\hat{f}(x)$ is bounded on the interval $(0 \leqq x < 1)$. Now we have

$$\hat{f}\left(1-\frac{1}{n}\right)-\hat{s}_n=\sum_{k=1}^{\infty} \hat{a}_k \left(1-\frac{1}{n}\right)^k-\hat{s}_n,$$

where $\hat{s}_k-\hat{s}_{k-1}=\hat{a}_k$, and

$$\hat{f}\left(1-\frac{1}{n}\right)-\hat{s}_n \leqq \sum_{k=1}^{n} |\hat{a}_k| \, |\left(1-\frac{1}{n}\right)^k-1|+\frac{C\left(1-\frac{1}{n}\right)^n}{n} \sum_{k=1}^{\infty} \left(1-\frac{1}{n}\right)^k$$

$$\leqq \frac{1}{n} \sum_{k=1}^{n} k|\hat{a}_k|+C=O(1).$$

From this it follows that $\hat{s}$ is a bounded sequence and hence s is a bounded sequence. This completes our proof which for the most part is derived from standard techniques, see (1) for example.

It is well known that $(C, 1)$ and the Abel method limit the same set of bounded sequences, see (1). All we need do is to show that the class of bounded sequences limited by $(C, 1)$ intersects $\bar{E}$ in the convergent sequences and we shall have proved that $\bar{E}$ is a Tauberian condition for both $(C, 1)$ and Abel summability.

THEOREM 2. The set $\bar{E}$ is a Tauberian condition for both $(C, 1)$ and Abel summability.

PROOF: * We shall in fact show that if $\{s_n\}$ is $(C, 1)$ limitable and $\{s_n\} \in \bar{E}$ then $\{s_n\}$ is convergent.

Since $\{s_n\} \in \bar{E}$, for every $\varepsilon>0$ there exists a $C_\varepsilon \geqq 1$ and sequence $\{s_n'\}$ such that

$$|s_n'-s_{n-1}'| \leqq \frac{C_\varepsilon}{n}, \quad |s_n-s_n'| \leqq \varepsilon, \quad (n=1, 2, \ldots).$$

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* I thank Professor J. Korevaar for this simple proof of Theorem 2.

Let $S_n = s_1 + \ldots + s_n$. We may assume without loss of generality that the Cesàro limit of $\{s_n\}$ is zero. Hence, for every $\delta > 0$, there exists an $n_0(\delta)$ such that $|S_n| < \delta n$, $n > n_0(\delta)$. Now for $0 < h \leqq n$,

$$
\begin{aligned}
|s_n| &= \left| \frac{S_{n+h} - S_n}{h} - \frac{(s_{n+1} - s_n) + \ldots + (s_{n+h} - s_n)}{h} \right| \\
&\leqq \frac{\delta(n+h+n)}{h} + \frac{|s'_{n+1} - s'_n| + \ldots + |s'_{n+h} - s'_n|}{h} \\
&\qquad + \frac{|s'_{n+1} - s_{n+1}| + |s'_n - s_n| + \ldots + |s'_{n+h} - s_{n+h}| + |s'_n - s_n|}{h} \\
&\leqq 3\,\delta \frac{n}{h} + c_\varepsilon \frac{(1 + \ldots + h)}{nh} + 2\frac{h\varepsilon}{h} \\
&\leqq 3\,\delta \frac{n}{h} + c_\varepsilon \frac{h}{n} + 2\varepsilon.
\end{aligned}
$$

After choosing ε, we may take $\delta = \varepsilon^2 / C_\varepsilon$ and $h = [(\varepsilon / C_\varepsilon) n]$ and it follows that $|s_n| < 7\varepsilon$, $n > n_0$.

2. Since $\bar{E}$ includes all of the convergent sequences, it is in some ways a more satisfactory Tauberian condition than E. We shall now give an example of a non convergent sequence in $\bar{E} \backslash E$. This example will be a sequence s satisfying $s_n - s_{n-1} = O(f(n)/n)$, lim sup $f(n) = \infty$, lim inf $f(n) = 1$.

Suppose that $s_n = s(n)$ has been defined for $n \leqq m(k) = m(k, 0)$ and that $s(m(k, 0)) = O$. We proceed to define $s(n)$ and the indices

$$m(k, 0), m(k, 1) \ldots m(k, 2r) = m(k+1, 0).$$

Let $s(n) - s(n-1) = 1/n$, $m(k, 0) \leqq n - 1 < m(k, 1)$ provided that $s(n-1) < \frac{1}{2} - 1/n$; otherwise, let $s(n) = s(m(k, 1)) = \frac{1}{2}$. Let $s(n) - s(n-1) = -1/n$ $m(k, 1) \leqq n - 1 < m(k, 2)$ provided that $s(n-1) > 1/n$; otherwise, let $s(n) = s(m(k, 2)) = 0$.

In general let, $s(n) - s(n-1) = 10^p/n$, $m(k, 2p) \leqq n - 1 < m(k, 2p+1)$ provided $s(n-1) < 2^{-(p+1)} - 10^p/n$; otherwise let $s(n) = s(m(k, 2p+1)) = 2^{-(p+1)}$. Let $s(n) - s(n-1) = -10^p/n$, $m(k, 2p+1) \leqq n - 1 < m(k, 2p+2)$ provided that $s(n-1) > 10^p/n$; otherwise let $s(n) = s(m(k, 2p+2)) = O$.

This process is to continue until $10^p/n > 2^{-(p+1)}$ or $p = k^2$ whichever occurs first. In this case, set $m(k, 2p) = m(k, 2r) = m(k+1, 0)$ and the process begins again with $s(m(k+1)) = 0$.

We can construct $\{x(n)\}$ so that $x(n) = 0$ if $|s(n) - s(n-1)| \geqq 10^p/n$ and $x(n) = s(n)$ if $|s(n) - s(n-1)| < 10^p/n$. For this sequence we clearly have $|x(n) - x(n-1)| < 10^p/n$ and sup $|s(n) - x(n)| < 2^{-(p+2)}$ so that $s \in \bar{E}$.

On the other hand, $m(k, 2p+3) - m(k, 2p+2)$ is approximately

$$m(k, 2p+2) \times 10^{-(p+1)} \times 2^{-(p+2)}$$

and if $|x(n)-x(n-1)| \leqq 10^p/n$ then $|x(m(k, 2p+3))-x(m(k, 2p+2))|$ is less than

$$m(k, 2p+2) \times 10^{-(p+1)} \times 2^{-(p+2)} \times \frac{10^p}{m(k, 2p+2)} \text{ or } \frac{1}{10} \times 2^{-(p+2)}$$

and sup $|s(n)-x(n)| > 2^{-(p+5)}/10$ and $s \notin E$.

REFERENCES

1. Zeller, K. and W. Beekman – "Theorie der Limitierungsverfahren", Second Ed. Springer Verlag, New York, Berlin (1970).
2. Lorentz, G. G. – "Direct theorems on methods of summability", Canad. J. Math. 1, 305–319 (1949).
3. Petersen, G. M. – "Factor sequences for summability matrices", Math. Z. 112, 389–392 (1969).
4. Petersen, G. M. – "Factor sequences and their algebras", Jber. Deutsch. Math.-Verein. 74, 182–188 (1973).